POLYNOMIAL-TIME APPROACHES TO FUZZINESS IN RATIO LABELLED GENERAL GRAPHS

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Abstract: In classical fuzzy graph theory, fuzziness is introduced by freely assigning membership values to vertices and edges, which makes every crisp graph trivially fuzzy. To avoid this limitation, we propose ratio labeling (RL), a structured labelling scheme in which vertex and edge memberships are linked by a fixed functional relationship. This approach provides a meaningful and non-trivial criterion for deciding when a graph can truly be considered fuzzy. We derive necessary and sufficient conditions under which a crisp graph becomes a fuzzy graph under ratio labelling and characterize the class of general graphs that admit fuzziness in this framework. We also present deterministic polynomial-time algorithms to label the vertices and edges of a general graph and to determine whether a given graph admits fuzziness under ratio labelling. These results offer an efficient and systematic method for studying fuzziness in general graphs and highlight the potential of ratio labelling for applications in connectivity analysis.

Keywords and Phrases: Fuzzy graph, Ratio Labelling, Ratio Labelled fuzzy graph.

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1. Introduction

Many real-world systems exhibit some degree of uncertainty. Fuzzy graph theory provides a way to model and analyze these systems more accurately than

traditional graph theory. Fuzzy relations, introduced by Zadeh in 1965, paved the way for the concept of fuzzy graphs. Kaufmann's initial work in 1973 was built upon these relations, but it was Rosenfeld's 1975 contributions, specifically his exploration of fuzzy relations on fuzzy sets, that significantly advanced the field. Rosenfeld's more formal definition of fuzzy graphs, which built upon Kaufmann's initial work, provided the necessary framework for the field's advancement. His introduction of concepts like fuzzy paths, cycles, and connectedness enabled further research into diverse fuzzy graph types, including intuitionistic and bipolar fuzzy graphs. Musavarah Sarwar et al. discussed decision making approach based on competition graphs and extended TOPSIS method under Bipolar Fuzzy Environment [9]. Several key developments have marked recent progress in fuzzy graph theory. The intuitionistic fuzzy organizational and neural network models, intuitionistic fuzzy neurons in medical diagnosis, intuitionistic fuzzy digraphs in vulnerability assessment of gas pipeline networks, and intuitionistic fuzzy digraphs in travel time are presented as examples of intuitionistic fuzzy digraphs in decision support system by Muhammad Akram et al [10]. Subsequent research has explored the following diverse applications: Koam et al. applied fuzzy graphs to crime detection, Arya Sebastian et al. discussed generalized fuzzy graph connectivity parameters with application to human trafficking, Bera et al. examined intuitionistic fuzzy graph connectivity for bank mergers, and Razzaque et al. used t-intuitionistic fuzzy graphs for complex relationship analysis [1, 4, 6, 14].

Gani and Rajalakshmi introduced fuzzy graph labeling in 2012 [11, 12]. Cherin and Asha analyzed hamiltonian fuzzy anti-magic labeling [5]. Graceful labeling for complete bipartite fuzzy graphs given by R. Jebesty Shajila et al. [7]. A new type of labelling named fuzzy range labelling in graph theory is introduced by S. Ramya et al. to handle uncertainty in graph labelling areas. The fuzzy value of an edge of fuzzy range labelling is the difference between the values of its incident vertices, aiming for distinct, meaningful labellings to model uncertainty [13]. Fuzzy magic labelling and fuzzy bi-magic labelling are some of the different types of labelling defined in fuzzy graphs. All these labelling methods assign labels to vertices without considering the structural parameters of the graph. This motivates us to define a labelling that takes the structure of the graph into account. Hence we introduced Ratio Labelling (RL) to analyze the fuzziness of graphs [2, 3]. Ratio labelling introduces fuzziness based on intrinsic graph parameters (such as vertex degrees), rather than assigning arbitrary membership values. This makes fuzziness meaningful and structurally driven. Unlike classical fuzzy graph models, where every crisp graph can be made fuzzy, ratio labelling allows fuzziness only in graphs with suitable structural properties. Also, RL may not have distinct membership

values as two or more vertices may have a same structural property.

In this paper, the scope of the investigation is extended to general graphs. The central problem addressed in this work is the following:

Given a general graph $G = (V, E)$, determining whether the given general graph satisfies the admissibility condition of fuzziness using RL. Further, necessary and sufficient conditions are established for a general graph to admit fuzziness under RL, providing a non-trivial characterization of fuzzy graphs.The main objective is to find algorithms to assign membership values to the vertices and edges and to verify fuzziness of a general graph using RL. Three deterministic algorithms with polynomial time are developed to label and to identify fuzziness in general graphs using the proposed RL scheme.

Since this labelling process does not alter the vertex set or the edge set, all structural properties of the graph remain unchanged. In particular, the order, size, degree sequence, and connectivity of the graph are preserved. Moreover, if a graph admits ratio labelling, then any graph isomorphic to it also admits ratio labelling. Hence, ratio labelling satisfies the invariant property and depends only on the underlying structure of the graph.

In a social network, the relationships between people can be established using RL. Communication patterns among individuals vary over situations and evolve over time, making crisp representations of interactions inadequate. Fuzzy graph models address this limitation by assigning graded membership values to relationships; however, arbitrary membership assignments fail to capture the relative strength of interactions. RL provides a structured mechanism to model such variability by defining vertex and edge memberships through ratio-based relationships, thereby ensuring that uncertainty in communication is represented in a consistent, interpretable, and mathematically constrained manner. The communication between group of friends in a social media on two particular situation and time are interpreted as a graph with vertices representing people and communication between the individuals as edges. Such a situation analyzed using RL for admittance of fuzziness [8]. The graph that admits fuzziness under RL ensures a strong connectivity condition among vertices.

Section 2 contains preliminaries and Section 3 discusses the characterization theorem for admittance of fuzziness under RL. Section 4 gives algorithms for identifying fuzziness in general graphs under RL using the characterization theorem discussed in Section 3. In addition, Section 4 gives the proof of the correctness of the algorithm and presents the time complexity of the algorithms.

2. Basic Concepts

A fuzzy graph $G : (\sigma, \mu)$ is a pair of functions, $\sigma : V \to [0, 1]$ and $\mu : V \times V \to$

$[0, 1]$, *where for all* $x, y \in V$,

$$\mu(x, y) \leq \sigma(x) \wedge \sigma(y)$$

where $\wedge$ stands for minimum, and

$$\sigma^* = supp(\sigma) = \{u \in S : \sigma(u) > 0\}.\mu^* = supp(\mu) = \{(u, v) \in S \times S : \mu(u, v) > 0\}.$$

If $\mu(x, y) > 0$ then x and y are called neighbours, x and y are said to lie on the same edge e. The neighbourhood of a vertex $v \in S$ is a set of all vertices that are neighbours of v denoted by $N(v)$. A fuzzy graph (σ', μ') is a fuzzy subgraph or a partial fuzzy subgraph of (σ, μ) if $\sigma' \subseteq \sigma$ and $\mu' \subseteq \mu$; that is, if $\sigma'(u) \leq \sigma(u)$ for every $u \in S$ and $\mu'(e) \leq \mu(e)$ for every $e \in E$. A fuzzy graph (σ', μ') is a fuzzy spanning subgraph (σ, μ) if $\sigma' = \sigma$ and $\mu' \subseteq \mu$; that is, if $\sigma'(u) = \sigma(u)$ for every $u \in S$ and $\mu'(e) \leq \mu(e)$ for every $e \in E$. Let $G : (\sigma, \mu)$ be a fuzzy graph.
The degree of the vertex v of a fuzzy graph G is defined as

$$deg_G(v) = \sum_{u \neq v} \mu(u, v).$$

In a fuzzy graph G the minimum degree $\delta(G)$, and maximum degree $\Delta(G)$, are defined as follows.

$$\delta(G) = min\{deg_G(u) : for\ all\ u \in V\}$$

and

$$\Delta(G) = max\{deg_G(u) : for\ all\ u \in V\}.$$

The order of a fuzzy graph, $G(\sigma, \mu)$ is defined as $O(G) = \sum_{u \in V} \sigma(u)$. The size of a fuzzy graph $G(\sigma, \mu)$ is defined as $S(G) = \sum_{(u,v) \in E} \mu(u, v)$. A fuzzy graph G is said to be regular for a positive real number k if $deg_G(u) = k$, for all $u \in V$. In this case, G is called $k-$regular fuzzy graph.

A vertex u, is said to be an isolated vertex if $\mu(u, v) = 0$ for all $u \neq v$.

3. Main Results

Ratio labelling has an clear way in expressing the admissibility of fuzziness in graphs. It does not make every graph fuzzy. Treating all graphs as fuzzy may not provide any meaningful information about their structure. However, some special classes of graphs, such as regular and complete graphs, are fuzzy under ratio labelling [2]. This shows that ratio labelling introduces fuzziness in well-structured graphs. Motivated by this, we studied general graphs under ratio labelling and

identified the structural properties that allow or restrict fuzziness. This leads to a condition based on the degrees of the end vertices of edges. The main objective of this section is to characterize general graphs that admits fuzziness under ratio labelling.

An algorithm for labelling the vertices and edges of a crisp graph under RL is presented.

Definition 3.1. *Let $G = (V, E)$ be a simple connected graph. The functions, $\sigma : V \to [0, 1]$ and $\mu : E \to [0, 1]$ that label the vertices and edges of G, are defined as*

$$\sigma(v) = \frac{|N(v)|}{|E|} \tag{1}$$

$$\mu(u, v) = \frac{max_{(u,v) \in E}[\sigma(u), \sigma(v)]}{\sum_{v \in V} \sigma(v)} \tag{2}$$

and is called ratio labelling of G. The graph G that is a fuzzy graph due to ratio labelling are called Ratio Labelled Fuzzy Graphs (RLFG).

An example and a counter example to depict ratio labelling are given below.

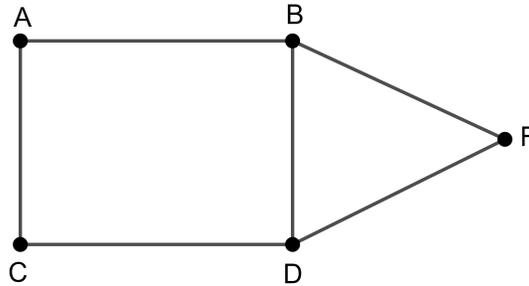**Example 3.2.** For a graph G in Figure 1,

Figure 1: A general graph that is a fuzzy graph under ratio labelling

the nodes are labelled using (1) as,

$$\sigma(A) = \sigma(C) = \sigma(F) = \frac{|N(v)|}{|E|} = \frac{2}{6}; \sigma(B) = \sigma(D) = \frac{3}{6}$$

and the edges of G are labelled using(2) as,

$$\mu(A, B) = \frac{max[\sigma(A), \sigma(B)]}{\sum_{v \in V} \sigma(v)} = \frac{max\{\frac{2}{6}, \frac{3}{6}\}}{3 \times \frac{2}{6} + 2 \times \frac{3}{6}} = \frac{\frac{3}{6}}{2} = \frac{1.5}{6};$$

$$\mu(C, D) = \frac{max\{\frac{2}{6}, \frac{3}{6}\}}{3 \times \frac{2}{6} + 2 \times \frac{3}{6}} = \frac{\frac{3}{6}}{2} = \frac{1.5}{6};$$

$$\mu(A, C) = \frac{max\{\frac{2}{6}, \frac{2}{6}\}}{3 \times \frac{2}{6} + 2 \times \frac{3}{6}} = \frac{\frac{2}{6}}{2} = \frac{1}{6};$$

$$\mu(B, D) = \frac{max\{\frac{3}{6}, \frac{3}{6}\}}{3 \times \frac{2}{6} + 2 \times \frac{3}{6}} = \frac{\frac{3}{6}}{2} = \frac{1.5}{6};$$

$$\mu(B, F) = \mu(D, F) = \frac{max\{\frac{2}{6}, \frac{3}{6}\}}{3 \times \frac{2}{6} + 2 \times \frac{3}{6}} = \frac{\frac{3}{6}}{2} = \frac{1.5}{6};$$

Here, $\mu(u, v) \leq \sigma(u) \wedge \sigma(v)$ for all $(u, v) \in E$. Hence G is a fuzzy graph under ratio labeling.

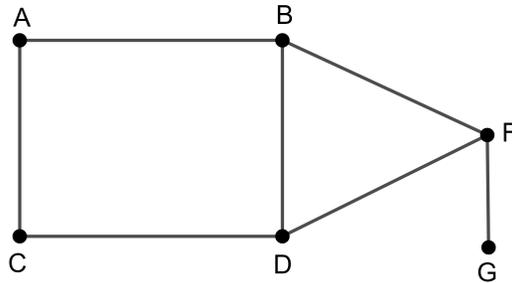**Example 3.3.** For a general graph G_1, in Figure 2,

Figure 2: A general graph that is not fuzzy under ratio labelling

using (1) the nodes are labelled as,

$$\sigma(A) = \sigma(C) = \frac{|N(v)|}{|E|} = \frac{2}{7}; \sigma(B) = \sigma(D) = \sigma(F) = \frac{3}{7}; \sigma(G) = \frac{1}{7}$$

and the edges of G are labelled using(2) as,

$$\mu(A, C) = \frac{max[\sigma(A), \sigma(C)]}{\sum_{v \in V} \sigma(v)} = \frac{max\{\frac{2}{7}, \frac{2}{7}\}}{2 \times \frac{2}{7} + 3 \times \frac{3}{7} + \frac{1}{7}} = \frac{\frac{2}{7}}{2} = \frac{1}{7};$$

$$\mu(A, B) = \mu(C, D) = \frac{max\{\frac{2}{7}, \frac{3}{7}\}}{2 \times \frac{2}{7} + 3 \times \frac{3}{7} + \frac{1}{7}} = \frac{\frac{3}{7}}{2} = \frac{1.5}{7};$$

$$\mu(B, D) = \mu(B, F) = \mu(D, F) = \frac{max\{\frac{3}{7}, \frac{3}{7}\}}{2 \times \frac{2}{7} + 3 \times \frac{3}{7} + \frac{1}{7}} = \frac{\frac{3}{7}}{2} = \frac{1.5}{7};$$

$$\mu(F,G) = \frac{max\{\frac{3}{7}, \frac{1}{7}\}}{2 \times \frac{2}{7} + 3 \times \frac{3}{7} + \frac{1}{7}} = \frac{\frac{3}{7}}{2} = \frac{1.5}{7};$$

Here, $\mu(F,G) = \frac{1.5}{7} > \frac{1}{7} = \sigma(F) \wedge \sigma(G)$ for $(F,G) \in E$.
Hence, G_1 is not a fuzzy graph under RL.

Remark 3.4. *Here, we analyzed two simple connected graphs for the admittance of fuzziness using RL. G satisfies the condition $\mu(u,v) \leq \sigma(u) \wedge \sigma(v)$ for all $(u,v) \in E$ but, in G_1 the edge FG violates the condition $\mu(u,v) \leq \sigma(u) \wedge \sigma(v)$. This indicates that there is a lack of communication between G and the other vertices.*

Remark 3.5. $\sum_{v \in V} \sigma(v) = \sum_{v \in V} \frac{|N(v)|}{|E|} = \frac{1}{|E|} \sum_{v \in V} |N(v)| = \frac{2|E|}{|E|} = 2.$
Hence, the order of a ratio labelled fuzzy graph is always two.

3.1. Ratio Labelling Algorithm for General Graphs

An algorithm RL(N,E,A) for labelling the vertices and edges of a general graph using RL is discussed in this section. Using adjacency matrix of the given graph, the vertices and edges are labelled using the definitions of σ and μ given in (1) and (2) respectively. Here, N and E represents the number of vertices and edges of the given graph, and A is its adjacency matrix.

Algorithm: RL(N,E,A)

> **Step 1.** *Read N, E*

> **Step 2.** *Read $A[N, N]$*

> **Step 3.** *Repeat for $I = 1$ to N*
> *$S[I] = 0$ [Initialize array $S[I]$]*
> *[End of loop]*

> **Step 4.** *(i)Repeat for $I = 1$ to N*
> *(ii)Repeat for $J = 1$ to N*
> *(iii)$S[I] = S[I] + A[I, J]$ [Row Sum]*
> *[End of inner loop J]*
> *[End of outer loop I]*

> **Step 5.** *(i)Repeat for $I = 1$ to N*
> *$\sigma[I] = \frac{S[I]}{E}$*
> *[End of loop I]*

> **Step 6.** *(i) Repeat for $I = 1$ to N*
> *(ii) Repeat for $J = 1$ to N*

(iii) If $A[I, J] \neq 0$ then

(iv) If $\sigma[I] > \sigma[J]$ then Set $X = \sigma[I]$ Else Set $X = \sigma[J]$

$\mu[I, J] = \frac{X}{2}$

[End of IF Structure]

[End of IF Structure]

[End of inner loop J]

[End of outer loop I]

Step 7. *Exit*

Theorem 3.6. *Let G be a general graph with N vertices and E edges. The labelling of vertices and edges of a general graph using RL is polynomially solvable with time complexity $O(N^2)$.*

Proof. Consider a graph G with N vertices and E edges. Let A be an adjacency matrix of G. Let the total time of execution of Steps 1, 2 be t. The time complexity of Step 3 in the algorithm is $O(N)$. Step 4 executes a for loop with an assignment statement $S[I] = S[I] + A[I, J]$, for N times. Let the time of execution of Step 4(iii) be c. The time complexity of Step 4 in the algorithm is $O(N^2)$. Now, Step 5 encounters a loop that is executed in N times. The time complexity for Step 5 is $O(N)$. Similarly, Step 6 is executed with time complexity $O(N^2)$. The time complexity of the above algorithm is $O(t + N + N^2 + N + N^2)$ i.e. $O(N^2)$ which is a polynomial time. Hence, for a given graph with N vertices and E edges, the problem can be solved in a polynomial time.

Characterization of a General Graph for Admittance of Fuzziness using RL

Example 3.2 and 3.3 of section 3, shows that not all graphs admit fuzziness under ratio labeling, hence it is necessary to examine the structural properties of general graphs that either promote or preclude fuzziness. Accordingly, general graphs are analyzed and a characterizing condition , $deg\,(u) \leq 2 \times deg\,(v)$ whenever $deg\,(v) \leq deg\,(u)$, is derived for every edge $(u, v) \in E$. The necessary condition discusses three cases for the edges $(u, v) \in E$ with $(i)\,deg\,(v) = deg\,(u)$, $(ii)\,deg\,(v) < deg\,(u)$ and $(iii)\,deg\,(v) > deg\,(u)$.

Theorem 3.7. *Let $G(V, E)$ be any connected simple graph. G is a RLFG iff for every edge (u, v) of G, $deg\,(u) \leq 2 \times deg\,(v)$ whenever $deg\,(v) \leq deg\,(u)$, that is, twice the minimum degree of the end vertices of an edge is greater than or equal to degree of the other end.*

Proof. Let G be a connected simple graph with n vertices and e edges. Let us examine the admittance of fuzziness using $\mu(u, v) \leq \sigma(u) \wedge \sigma(v)$ for all $(u, v) \in E$. In

any graph G, the vertex v of an edge (u,v) be either $(i)\ deg(v) = deg(u)$, or $(ii)\ deg(v) < deg(u)$ and $(iii)\ deg(v) > deg(u)$.

Case (i)

Let $deg(v) = deg(u) = k$ .Then by RL, $\sigma(v) = \sigma(u) = \frac{k}{e}$, $\mu(u,v) = \frac{k}{2e}$. Hence,

$$\mu(u,v) = \frac{k}{2e} < \sigma(u) \wedge \sigma(v).$$

(1)

Case (ii)

Let $deg(v) < deg(u)$.

Let $deg(v) = a; deg(u) = b$, hence $a < b$.

Then by RL, $\sigma(v) = \frac{a}{e}; \sigma(u) = \frac{b}{e}, \mu(u,v) = \frac{b}{2e}$

    Case (a) Suppose $2 \times deg(v) = deg(u) \Rightarrow 2a = b$

$$Hence,\ \mu(u,v) = \frac{b}{2e} = \frac{2a}{2e} = \frac{a}{e} = \sigma(u) \wedge \sigma(v).$$

(2)

    Case (b) Suppose $2 \times deg(v) > deg(u) \Rightarrow 2a > b$

$$Hence,\ \mu(u,v) = \frac{b}{2e} < \frac{2a}{2e} = \frac{a}{e} = \sigma(u) \wedge \sigma(v).$$

(3)

    Case (c) Suppose $2 \times deg(v) < deg(u) \Rightarrow 2a < b$. Hence, $\mu(u,v) = \frac{b}{2e} > \frac{2a}{2e} > \frac{a}{e} = \sigma(u) \wedge \sigma(v)$. Here, fuzziness fails for $2 \times deg(v) < deg(u)$

From (2) and (3), $\mu(u,v) \leq \sigma(u) \wedge \sigma(v)\ for\ 2 \times deg(v) \geq deg(u)$ and by case(c), G is not a RLFG for $2 \times deg(v) < deg(u)$

Hence, the condition for G to be a RLFG for $deg(v) < deg(u)$, is that $2 \times deg(v) \geq deg(u)$ i.e., $deg(u) \leq 2 \times deg(v)$

Case (iii)

Let $deg(v) > deg(u)$. Here, degree of u is minimum. So, the condition is restated as $2 \times deg(u) \geq deg(v)$. Let us prove that G is a RLFG when $2 \times deg(u) \geq deg(v)$.

Let $deg(v) = a; deg(u) = b$, hence $a > b$.

Then by RL $\sigma(v) = \frac{a}{e}; \sigma(u) = \frac{b}{e}, \mu(u,v) = \frac{a}{2e}$

Case (a) Suppose $2 \times deg(u) = deg(v) \Rightarrow 2b = a$

$$Hence,\ \mu(u,v) = \frac{a}{2e} = \frac{2b}{2e} = \frac{b}{e} = \sigma(u) \wedge \sigma(v).$$

(4)

Case (b) Suppose $2 \times deg(u) > deg(v) \Rightarrow 2b > a$

$$Hence,\ \mu(u,v) = \frac{a}{2e} < \frac{2b}{2e} = \frac{b}{e} = \sigma(u) \wedge \sigma(v).$$

(5)

Case (c) Suppose $2 \times deg\,(u) < deg\,(v) \Rightarrow 2b < a$ *Hence,* $\mu\,(u,v) = \frac{a}{2e} > \frac{2b}{2e} > \frac{b}{e} = \sigma\,(u) \wedge \sigma\,(v)$. Here, fuzziness fails for $2 \times deg(u) < deg(v)$

From (4) and (5), $\mu\,(u,v) \leq \sigma\,(u) \wedge \sigma\,(v)$ *for* $2 \times deg\,(u) \geq deg\,(v)$

And by case(c), of case(iii), G is not a RLFG for $2 \times deg\,(u) < deg\,(v)$

From case(i), (ii) and (iii), G is a RLFG for $2 \times deg(v) \geq deg(u)$ i.e., $deg(u) \leq 2 \times deg(v)$.

Conversely, let G be a RLFG. We need to prove, for every edge (u,v) of G, $deg\,(u) \leq 2 \times deg\,(v)$ whenever $deg\,(v) \leq deg\,(u)$.

On the contrary suppose, there exist an edge (u,v) of G such that $deg\,(u) > 2 \times deg\,(v)$ whenever $deg\,(v) \leq deg\,(u)$

Let $deg\,(v) = k$. Then $deg\,(u) > 2k$. Hence $\sigma(v) = \frac{k}{e}$ and $\sigma(u) > \frac{2k}{e}$

$\sigma(u) \wedge \sigma(v) = \frac{k}{e}$, but $\mu(u,v) > \frac{2k}{2e} = \frac{k}{e} = \sigma(u) \wedge \sigma(v)$, which contradicts the fuzziness of G. Hence, G is a fuzzy graph for every edge (u,v) of G, $deg\,(u) \leq 2 \times deg\,(v)$ whenever $deg\,(v) \leq deg\,(u)$.

The statement is verified.

4. Algorithmic approach to fuzziness in general graphs

As ratio labelling doesn't ensure fuzziness in all graphs, we intend to move in search of a condition that violates fuzziness. To label a vertex in a crisp graph, ratio labelling concerns about number of edges that incident on a particular vertex with respect to total number of edges in the graph. So, the condition that ensures or violates fuzziness will depend on the degree of the vertices. The result in Theorem 3.7 is equally stated as, whenever the degree of the end vertices of every edge (u,v) of a graph satisfies the condition,

$$deg(u) \leq 2 \times deg(v) \ and \ deg(v) \leq 2 \times deg(u)$$

then the respective crisp graph becomes fuzzy under ratio labelling. Equivalently, the crisp graph fails to be fuzzy whenever

$$deg(u) \geq 2 \times deg(v) \ or \ deg(v) \geq 2 \times deg(u)$$

In this section, we formulated two algorithms RLFG1, RLFG2 to identify the fuzzy graphs in a general graph. The first algorithm RLFG1(E, DEG, X) analyses the admissibility of fuzziness with an array of degrees of vertices DEG and number of edges E of a graph G. The second algorithm RLFG2(N, A, X) analyses the fuzziness with adjacency matrix A and number of vertices N of the graph G . The variable X denotes the counter variable in both the algorithms. Both the algorithms examines the above condition in crisp graphs to identify fuzziness.

Further, theorem 4.1 and 4.2 gives the time complexity of the two algorithms.

Accordingly, both the algorithms are polynomially solvable. The verification of the algorithms are given in section 4.2, 4.4.

4.1. Algorithm: RLFG1(E, DEG, X)

(Identification of a fuzzy graph using Ratio labelling)$RLFG1(E, DEG, X)$

Here E represents the total number edges of a graph, and DEG is a $E \times 2$ array that reads the degree of the end vertices of every edge of the given graph G. This algorithm finds the number of edges X, which violates the condition in Theorem 3.7.

Step 1. *Set $X = 0.[Initialize\ the\ counter]$*

Step 2. *Repeat for $J = 1$ to E*
If $(DEG[J, 1] > 2*DEG[J, 2])$ or $DEG[J, 2] > 2*DEG[J, 1]$ then :
$X = X\ +\ 1$
[End of IF Structure]
[End of loop]

Step 3. *If $X > 0$ then : print "Given graph is not a fuzzy graph"*
Else : print "Given graph is a fuzzy graph"
[End of If structure]

Step 4. *Exit*

Theorem 4.1. *Let G be a general graph with N vertices and E edges. The admittance of fuzziness under RL in a general graph is polynomially solvable with complexity $O(E)$.*

Proof. Consider a graph G with N vertices and E edges. Let DEG be a $E \times 2$ array that reads the degree of the end vertices of every edge of the given graph.

The algorithm $RLFG1$ encounters a 'for' loop in Step 2 with a conditional statement that executes an increment statement $X = X+1$, whenever $DEG[J, 1] > 2*DEG[J, 2])or(DEG[J, 2] > 2*DEG[J, 1]$, is true. Let the execution time of step 2 be c. As the for loop in step 2 is executed 'E' times, the statement $X = X + 1$ is executed a maximum of $E \times c$ times. Step 3 verifies whether $X > 0$ or $X = 0$ and gives the output with constant time, say k. Hence, the total execution time is $E \times c + k$. The number of edges in the graph changes the execution time. Hence, the overall time complexity of the above algorithm is $O(E)$, which is a polynomial time.

Hence, for a graph with E edges, the algorithm is polynomially solvable with time complexity $O(E)$.

4.2. Proof of correctness of RLFG1

Consider the graph G of Example 3.2, take $X = 0$ and DEG reads the degree of the end vertices of the edges AB, CD, BF, DF, AC, BD of G. For $J = 1$

2	3
2	3
3	2
3	2
2	2
3	3

Table 1: DEG of Graph G in Figure 1

$$DEG[1, 1] < 2 \times DEG[1, 2] \text{ and}$$
$$DEG[1, 2] < 2 \times DEG[1, 1]$$

For $J = 2$

$$DEG[2, 1] < 2 \times DEG[2, 2] \text{ and}$$
$$DEG[2, 2] < 2 \times DEG[2, 1]$$

For $J = 3$

$$DEG[3, 1] < 2 \times DEG[3, 2] \text{ and}$$
$$DEG[3, 2] < 2 \times DEG[3, 1]$$

For $J = 4$

$$DEG[4, 1] < 2 \times DEG[4, 2] \text{ and}$$
$$DEG[4, 2] < 2 \times DEG[4, 1]$$

For $J = 5$

$$DEG[5, 1] < 2 \times DEG[5, 2] \text{ and}$$
$$DEG[5, 2] < 2 \times DEG[5, 1]$$

For $J = 6$

$$DEG[6, 1] < 2 \times DEG[6, 2] \text{ and}$$
$$DEG[6, 2] < 2 \times DEG[6, 1]$$

2	3
3	3
3	1
3	3
2	3
2	2
3	3

Table 2: DEG of Graph G_1 in Figure 2

At the end, $X = 0$, and the graph is fuzzy. RLFG1 admits the result given in Example 3.2.

Consider the graph G_1 of Example 3.3, $X = 0$ and DEG reads the degree of the end vertices of the edges $AB, BD, FG, BF, CD, AC, DF$

$DEG[J, 1] < 2 * DEG[J, 2])$ or $DEG[J, 2] < 2 * DEG[J, 1]$ is true for $J = 1, 2, 4, 5, 6, 7$

But for $J = 3$,

$$DEG[3, 1] = 3, DEG[3, 2] = 1$$
$$DEG[3, 1] > 2 \times DEG[3, 2]$$

Hence, X is incremented, $X = 1$. At the end as $X > 0$, the given graph is not fuzzy. RLFG1 admits the result given in Example 3.3.

4.3. Algorithm: RLFG2(N, A, X)

This algorithm $RLFG2$ identifies whether the given general graph is a fuzzy graph or not using RL. Let G be a graph with N vertices and E edges. Identification is carried out using the adjacency matrix A of the given graph G of order $N \times N$. The row sum, $S[I]$ of the matrix A is calculated to find the degree of the vertices of the graph. An entry $A[I, J] \neq 0$ of the matrix indicates that there is an edge between the vertices I and J. This algorithm finds the number of edges X that violates the condition of Theorem 3.7 for $A[I, J] \neq 0$. When $X > 0$, the graph is not a fuzzy graph. It is an effective algorithm with polynomial time $O(N^2)$.

Step 1. *Read N*

Step 2. *Read $A[N, N]$*

Step 3. $X = 0$ *[Initialize X]*

Step 4. *Repeat for $I = 1$ to N*
$\quad\quad\quad S[I] = 0$ $\quad\quad\quad\quad$ *[Initialize array $S[I]$]*
$\quad\quad\quad$ *[End of loop]*

Step 5. *(i)Repeat for $I = 1$ to N*
$\quad\quad\quad$ *(ii)Repeat for $J = 1$ to N*
$\quad\quad\quad S[I] = S[I] + A[I, J]$ $\quad\quad$ *[Row Sum]*
$\quad\quad\quad$ *[End of inner loop J]*
$\quad\quad\quad$ *[End of outer loop I]*

Step 6. *(i) Repeat for $I = 1$ to N*
$\quad\quad\quad$ *(ii) Repeat for $J = 1$ to N*
$\quad\quad\quad$ *(iii) If $A[I, J] \neq 0$ then*
$\quad\quad\quad$ *(iv) If $S[I] > 2*S[J]$ or $S[J] > 2*S[I]$ then Set $X = X + 1$ and return*
$\quad\quad\quad$ *[End of IF Structure]*
$\quad\quad\quad$ *[End of IF Structure]*
$\quad\quad\quad$ *[End of inner loop J]*
$\quad\quad\quad$ *[End of outer loop I]*

Step 7. *If $X > 0$ then : print "Given graph is not a fuzzy graph"*
$\quad\quad\quad$ *Else : print "Given graph is a fuzzy graph"*
$\quad\quad\quad$ *[End of If structure]*

Step 8. *Exit*

Theorem 4.2. *Let G be a general graph with N vertices and E edges. The admittance of fuzziness under RL in a general graph is polynomially solvable with time complexity $O(N^2)$.*

Proof. Consider a graph G with N vertices and E edges. Let A be an adjacency matrix of G. The time complexity of Step 2 in the RLFG2 algorithm is $O(N^2)$. Let the total time of execution of Steps 3, 7 be t. Step 4 executes a for loop with an assignment statement $S[I] = 0$, for N times. Let the time of execution of Step 4(i) be c. Then Step 4 is executed in $N \times c$ time. Now, Step 5 encounters a nested loop that is executed in N^2 times. Let Step 5(iii) take a constant time, say k, to execute. Hence, Step 5, is executed in $N^2 \times k$ time. The time complexity for Step: 5 is $O(N^2)$. Similarly, Step 6 is executed with time complexity $O(N^2)$. The time complexity of the above algorithm is $O(N^2 + N^2 + N + t)$ i.e. $O(N^2)$ which is a polynomial time. Hence, for a given graph with N vertices, the algorithm is polynomially solvable with time complexity $O(N^2)$.

4.4. Proof of Correctness of Algorithm RLFG2

The proof of correctness of the algorithm is done with Examples 3.2 and 3.3 of section 3.

For example 3.2, Steps 1 and 2 read the number of vertices, $N = 5$ and array A. Table 3 gives the array A, the adjacency matrix of G. Step 3 Assigns the

Vertices	A	B	C	D	F
A	0	1	1	0	0
B	1	0	0	1	1
C	1	0	0	1	0
D	0	1	1	0	1
F	0	1	0	1	0

Table 3: Adjacency Matrix of graph G of Figure 1

counter $X = 0$

Step 4 assigns zero to row sum, of the matrix A, that is, $S[I] = 0$, for all I.

The row sum is calculated in Step 5. For $I = 1$, $J = 1\ to\ 5$, $S[1] = 0+1+1+0+0 = 2$

For $I = 2$, $J = 1\ to\ 5$, $S[2] = 1 + 0 + 0 + 1 + 1 = 3$

For $I = 3$, $J = 1\ to\ 5$, $S[3] = 1 + 0 + 0 + 1 + 0 = 2$

For $I = 4$, $J = 1\ to\ 5$, $S[4] = 0 + 1 + 1 + 0 + 1 = 3$

For $I = 5$, $J = 1\ to\ 5$, $S[5] = 0 + 1 + 0 + 1 + 0 = 2$

This row sum gives the degree of the vertices A, B, C, D, F.

Step: 6, verifies the condition of fuzziness and the counter X is incremented when it is violated for the adjacent vertices.

At the end, $X = 0$, hence given graph is a fuzzy graph. RLFG2 admits the result given in Example 3.2.

Vertices	A	B	C	D	F	G
A	0	1	1	0	0	0
B	1	0	0	1	1	0
C	1	0	0	1	0	0
D	0	1	1	0	1	0
F	0	1	0	1	0	1
G	0	0	0	0	1	0

Table 4: Adjacency Matrix of graph G_1 of Figure 2

For example 3.3, Step : 1 and 2 read the number of vertices, $N = 6$ and array A. Table 4 gives the array A, the adjacency matrix of G_1. Step 3 Assign the counter $X = 0$

Step 4 assigns zero to row sum, of the matrix A , that lis, $S[I] = 0$, for $I = 1$ to N

The row sum is calculated in Step 5

For $I = 1$, $J = 1$ to 6, $S[1] = 0 + 1 + 1 + 0 + 0 + 0 = 2$

For $I = 2$, $J = 1$ to 6, $S[2] = 1 + 0 + 0 + 1 + 1 + 0 = 3$

For $I = 3$, $J = 1$ to 6, $S[3] = 1 + 0 + 0 + 1 + 0 + 0 = 2$

For $I = 4$, $J = 1$ to 6, $S[4] = 0 + 1 + 1 + 0 + 1 + 0 = 3$

For $I = 5$, $J = 1$ to 6, $S[5] = 0 + 1 + 0 + 1 + 0 + 1 = 3$

For $I = 6$, $J = 1$ to 6, $S[6] = 0 + 0 + 0 + 0 + 1 + 0 = 1$

This row sum gives the degree of the vertices.

Step: 6 verifies the condition of fuzziness and the counter X is incremented when it is violated .

For the edge FG,

$$S[5] = 3, S[6] = 1 \text{ Since, } S[5] > 2 \times DEG[6], X \text{ is incremented as } X = 1.$$

After completion of the loop, as $X > 0$, the given graph declines to be fuzzy graph.RLFG2 admits the result given in Example 3.3.

5. Conclusion

This work establishes a clear criterion for determining fuzziness in general graphs under ratio labelling. By presenting two polynomial-time algorithms, the study shows that fuzzy graphs can be identified efficiently, even within large and complex graph structures. The results highlight that graphs admitting fuzziness under ratio labelling possess strong connectivity properties, underscoring their importance in network-based applications. These findings provide a foundational framework for further theoretical investigations and practical implementations. In particular, the study opens avenues for applying ratio-labelled fuzzy graphs in areas such as network design, neural networks, and telecommunication systems, as well as for exploring fuzzy tree t-spanners in general graphs.

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